Computer Science

CSC380: Principles of Data Science

Probability 1

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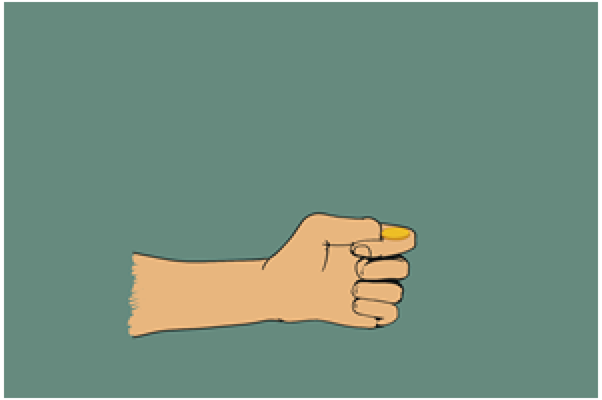
Outline

- What is probability?
- Events
- Calculating probabilities
- Set Theory
- Law of Total Probability

What is probability?

What is probability?

- Suppose I flip a coin, What is the probability it will come up heads? Most people say 50%, but why?



- “Nolan’s new movie is coming out next weekend! There’s a 100% chance you’re going to love it.”

Interpreting probabilities

Basically two different ways to interpret:

- Objective probability
  - based on logical analysis or long-run frequency of an event. It's derived from known facts, symmetry, or repeated experiments.
- Subjective probability
  - based on personal belief, opinion, or information about how likely an event is, especially when there's uncertainty or limited data.

Objective or Subjective?

r/AskReddit • 2 yr. ago

What statistically improbable thing happened to you?

• 2y ago

When I was a teenager I picked up a hitchhiker and then a few years later the same guy picked me up when I was walking after I ran out of gas. Never saw him before or after those two occasions.

• 2y ago

Got attacked by a robin in the morning, then attacked by a hawk 3 hours later. Weird day.

18K Award Share ...

It is a subjective probability: a belief based on their perception of how rare or meaningful the coincidence is, not a calculation based on statistical data.

Objective probability

- The probability of an event represents the long run proportion of the time the event occurs under repeated, controlled experimentation.
  - e.g. 0001110100111110100011O
- Famous experiments in history on coin tosses

Experimenter	# Tosses	# Heads	Half # Tosses
De Morgan	4092	2048	2046
Buffon	4040	2048	2020
Feller	10000	4979	5000
Pearson	24000	12012	12000

Subjective probability

- Probabilities aren't in the world itself; they're in our knowledge/beliefs about the world.
- Can assign a probability to the truth of any statement that I have a degree of belief about.

We will focus on objective probability in this class.

Outcome, Event and Probability

Outcome

- Outcome is a single result/observation of a random experiment.

- Example 1: You flip a coin once.
  - The outcomes are: "Heads" or "Tails”

- Example 2: You roll a 6-sided die.
  - The outcomes are: 1, or 2, or 3, or 4, or 5, or 6

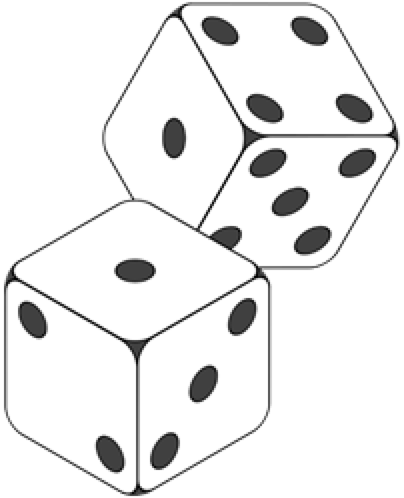
- Example 3: You tap "shuffle play" on your favorite Spotify playlist with 100 songs and the first song played.
  - The outcomes are: the 1st song in the list, or the 2nd song, ….

Random Events and Probability

Suppose we roll <u>two fair dice</u>…

Random Events and Probability

Suppose we roll <u>two fair dice</u>…



- What are the possible outcomes?
- What is the *probability* of rolling **even** numbers?
- What is the *probability* of having two numbers sum to 6?
- If one die rolls 1, then what is the probability of the second die also rolling 1?

*…this is a **random process**.*

How to formalize all these quantitatively?

The Sample Space

- The set of all possible outcomes of a random experiment is called the sample space, written as S.

- In math, the standard notation for a set is to write the individual members in curly braces:
  - S = {Outcome1, Outcome2, . . . , }

- Useful to visualize the sample space with an actual space.

The Sample Space

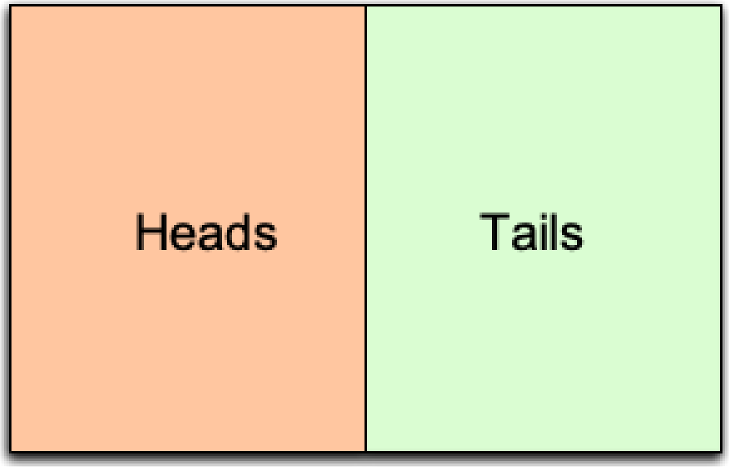
Probability very closely tied to area:

Figure: Visualization of a Sample Space

Examples of Sample Spaces

What's the sample space for a single coin flip?

- S = {Heads, Tails}



Examples of Sample Spaces

What is the sample space of rolling a die?

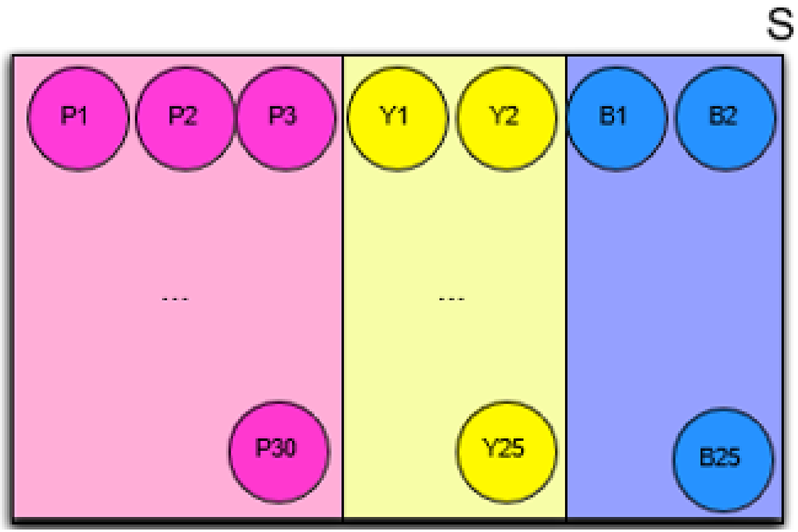
- S = {1, 2, 3, 4, 5, 6}

1	2	3
4	5	6

Examples of Sample Spaces

What is the sample space of drawing a ball out of a box containing 30 pink, 25 yellow, and 25 blue balls?

- S = {P1, P2, …, P30, Y1, …, Y25, B1, …, B25}



Examples of Sample Spaces

What's the sample space for...

- Randomly choosing a student from UA?
  - S = {Aarhus, Amaral, Balkan, . . . , Yao, Zielinski}
- Flipping two different coins?
  - S = {HH, HT, TH, TT}
- Flipping one coin twice?
  - S = {HH, HT, TH, TT}
- Observing the number of earthquakes in San Francisco in a particular year?
  - S = {0, 1, 2, 3, . . . }

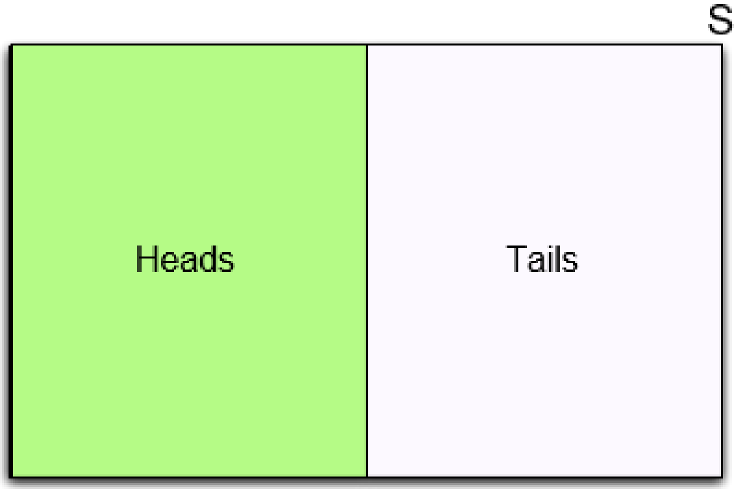
Events

- An event E is a **subset** of the sample space.
- An event E is a **set** of outcomes
- When we make a particular observation, it is either “in” E or not. Helpful to think about events as propositions (TRUE/FALSE).
  - The proposition is TRUE when the outcome is among the elements of the event set, and FALSE otherwise.
  - Is 4 in event E = {2, 4, 6}? ✅ YES → the **proposition is TRUE**
  - Is 4 in event F = {1, 3, 5}? ❌ NO → the **proposition is FALSE**

Examples of Events

What’s the event set corresponding to the following propositions?

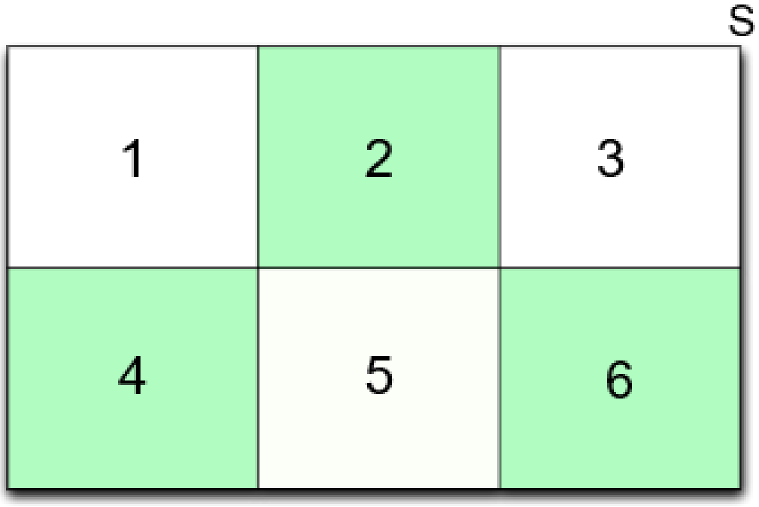
- “The coin comes up heads”
- E = {Heads}



Examples of Events

What's the event set corresponding to the following propositions?

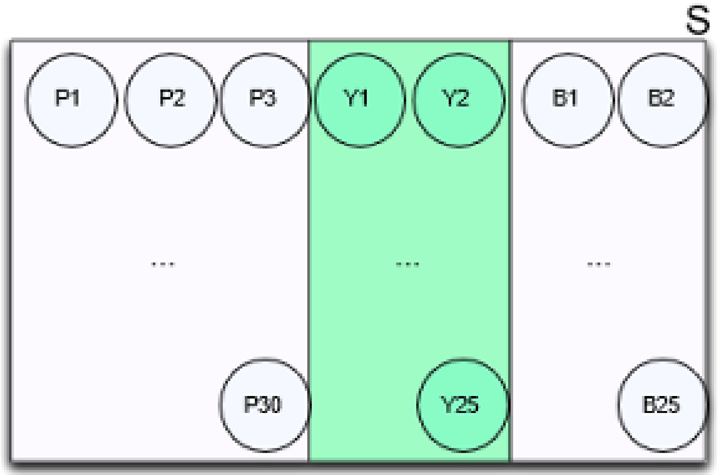
- "The die comes up an even number"
- E = {2, 4, 6}



Examples of Events

What's the event set corresponding to the following propositions?

- "A yellow ball is chosen"
- E = {Y1, Y2, . . . , Y25}



Special events

- The sample space (S) includes all possible outcomes.
  - If an event E = S, then no matter what outcome occurs it's always in E.
  - e.g., E = {Heads, Tails}

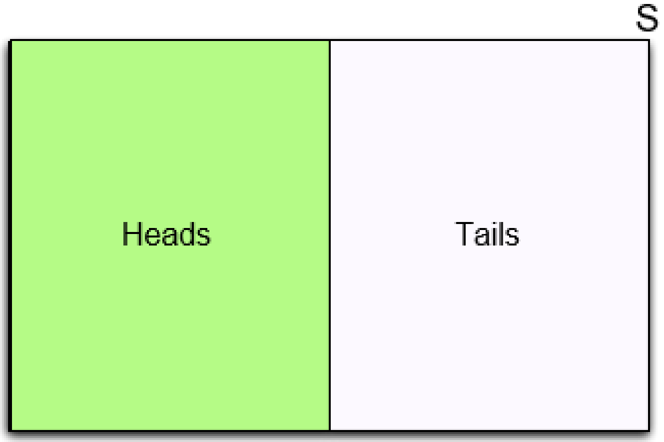
- The empty set ∅ is also an event
  - It is an event that never happens
  - e.g. “the die comes up 7”, E = {7}

Calculating Probabilities

Calculating probability

- We can think of the probability of an event E as its area, where S (sample space) always has a total area of 1.0

- So, the probability of E is the fraction of S that it takes up.

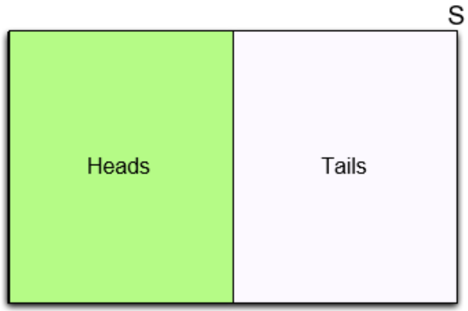


Calculating probability using symmetry

- If we have a sample space for which every outcome is equally likely, then we can find event probabilities easily.

- Classic probability model: if every outcome has the same "area", we can just count:

$$P(E) = \frac{\#\text{outcomes in } E}{\#\text{outcomes in } S}$$



Probability as Area

What is the probability of

$$P(E) = \frac{\#\text{outcomes in } E}{\#\text{outcomes in } S}$$

- Rolling a fair die and see an even number?
- E = {2,4,6}
- $P(E) = \frac{\#\{2,4,6\}}{\#\{1,2,3,4,5,6\}} = \frac{3}{6} = \frac{1}{2}$

1	2 P(2) = 1/6	3
4 P(4) = 1/6	5	6 P(6) = 1/6

P(S) = 1

P(Even) = P(2) + P(4) + P(6) = 3/6

Elementary events

- Notice that we can find the total probability of an event by breaking it into pieces and adding up the probabilities of the pieces:

  P(Even) = P(2) + P(4) + P(6)

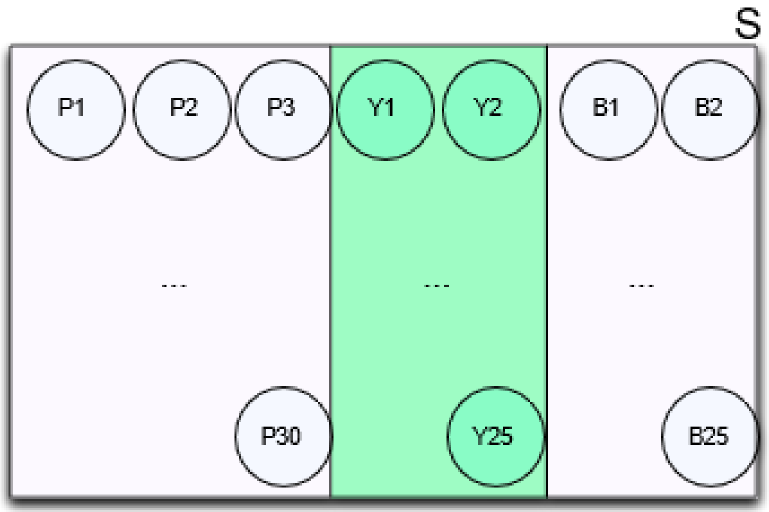
1	2 P(2) = 1/6	3
4 P(4) = 1/6	5	6 P(6) = 1/6

- These pieces are called ‘elementary events’: Events that correspond to exactly one outcome

Probability as Area

What is the probability of

- Selecting a yellow ball?
- E = {Y1, Y2, . . . , Y25}
- $P(E) = \frac{\# E}{\# S} = \frac{25}{30+25+25} = \frac{5}{16}$

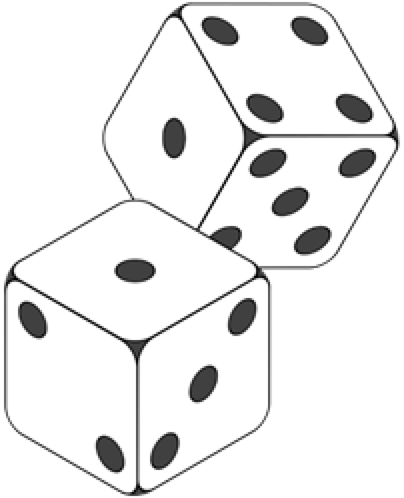


P(Yellow) = P(Y1) + ... + P(Y25) = 25 * (1/80)

Random Events and Probability

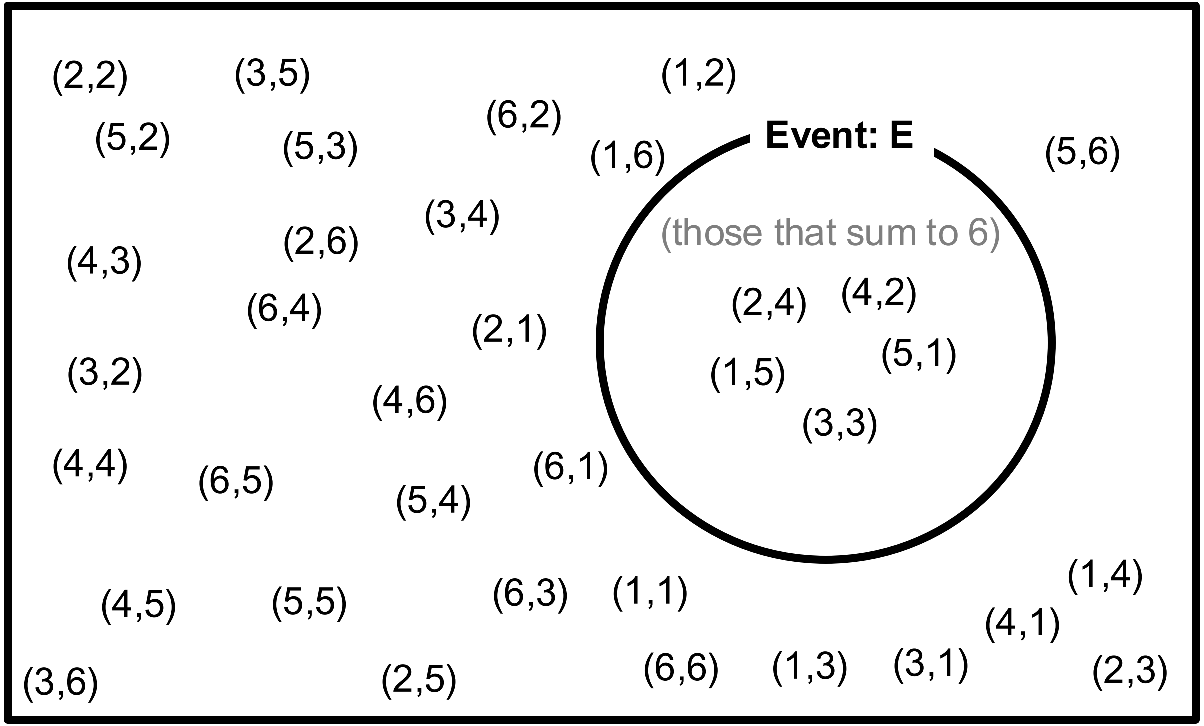
- Suppose we throw two fair dice
  - What is the sample space S (space of all possible outcomes)?

  Event E: the two dice's outcomes sum to 6
  - What is the size of E?
  - What is the probability of E?



Random Events and Probability

What is the probability of having two numbers sum to 6?



$$P(E) = \frac{\#\text{outcomes in } E}{\#\text{outcomes in } S}$$

S = $\{(a, b): a, b \in \{1, \ldots, 6\}\}$

Each outcome is equally likely

# of outcomes that sum to 6:
5

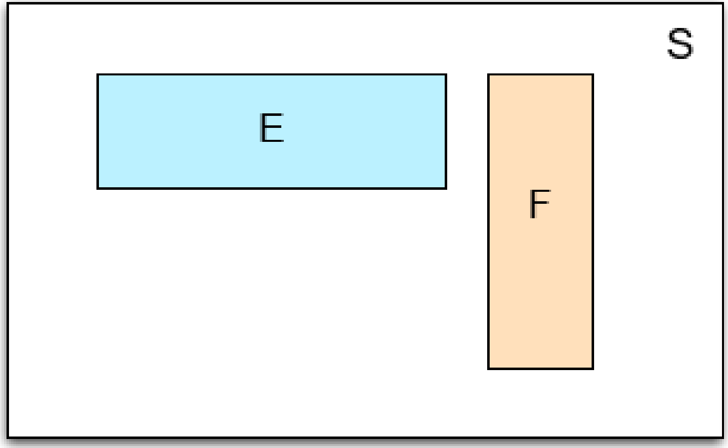
answer:
5/36 = 0.13888…

Disjoint events

- In general, breaking an event into *disjoint events* preserves the total probability
- **Disjoint**: E and F are *disjoint* if they cannot happen simultaneously,
- e.g. E = {2, 4}, F={1, 3, 5}

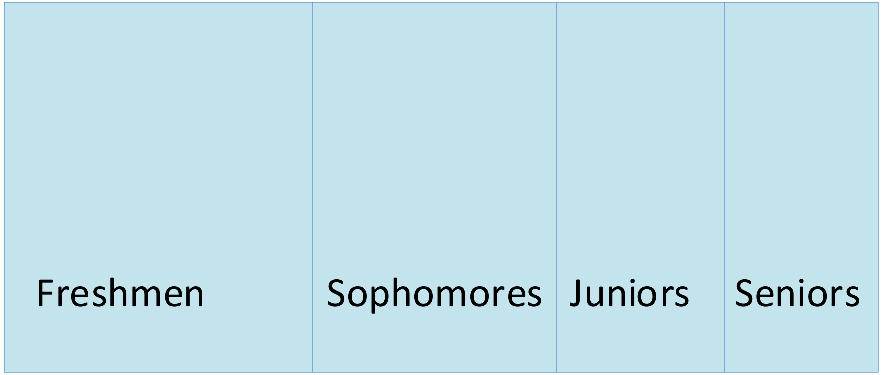
- In such cases,

$$P(E \text{ or } F) = P(E) + P(F)$$



Partition

- We say that disjoint events $E_1, \ldots, E_n$ form a *partition* of E if any outcome in E lies in exactly one E_i
- e.g.
  - {Fr.} E_1, {Soph.} E_2 form a partition of {Lower division} E
  - {Fr.}, {Soph.} {J.} {Sen.} form a partition of S (sample space)



Probability as Area

For disjoint events E, F:

$$P(E \text{ or } F) = P(E) + P(F)$$

If disjoint events $E_1, \ldots, E_n$ forms a partition of E :

$$P(E) = P(E_1) + P(E_2) + \cdots + P(E_n)$$

If disjoint events $E_1, \ldots, E_n$ forms a partition of S , for event F(law of total probability):

$$P(F) = P(E_1, F) + P(E_2, F) + \cdots + P(E_n, F)$$

Notation: $P(A, B)$ is a shorthand for $P(A \text{ and } B)$

Probability as Area

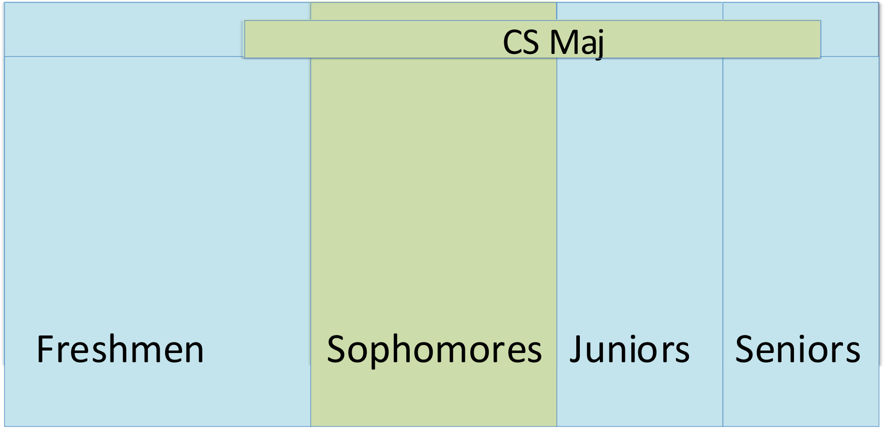
Examples

- $P(\text{CS}) = P(\text{Fr.}, \text{CS}) + P(\text{Soph.}, \text{CS}) + P(\text{J.}, \text{CS}) + P(\text{Sen.}, \text{CS})$

Notation: $P(A, B)$ is a shorthand for $P(A \text{ and } B)$

- $P(\text{Soph.}) = P(\text{CS}, \text{ Soph.}) + P(\text{nonCS}, \text{ Soph.})$

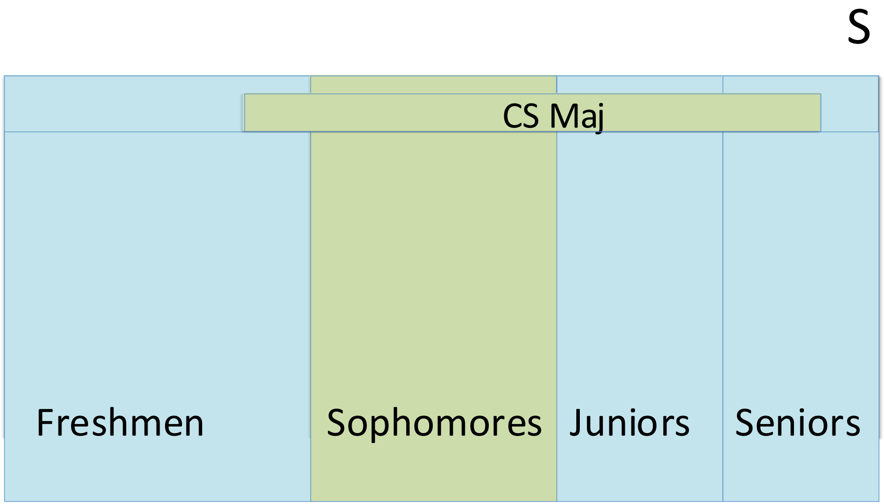
S



Probability as Area

If events E, F are non-disjoint, what is $P(E \text{ or } F)$?

- $P(\text{Soph.} \, or \, \text{CS})$?
- Note: events "Sophomore" and "CS Major" may overlap

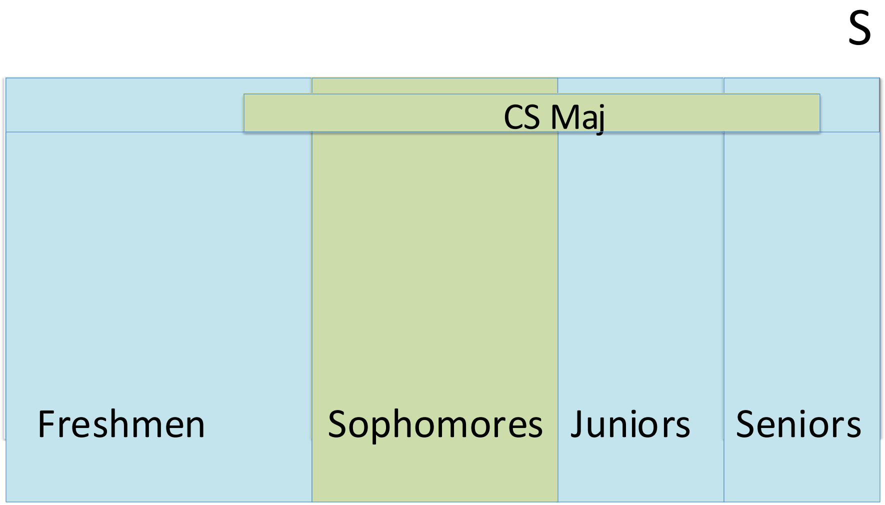


Probability as Area

- E = { Soph OR CS }

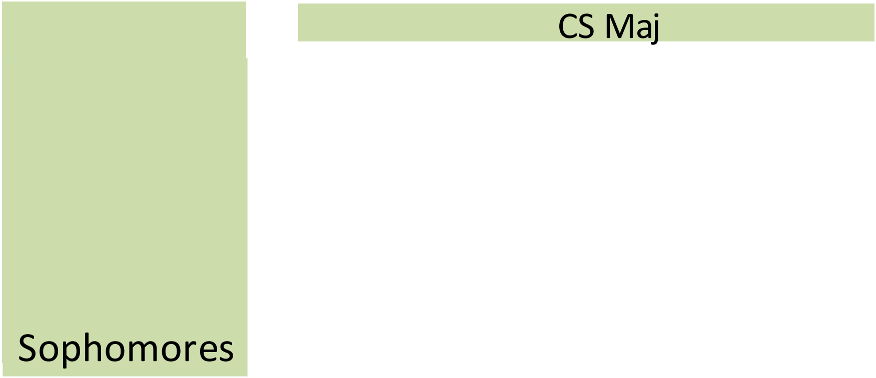
- Is P(E) = P(Soph) + P(CS)?
  - No

- Which one is larger?
  - Let's see..

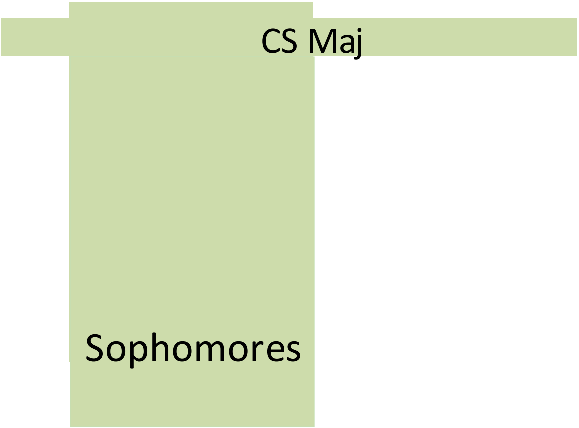


Probability as Area

P(Soph) + P(CS) =



P(Soph or CS) =



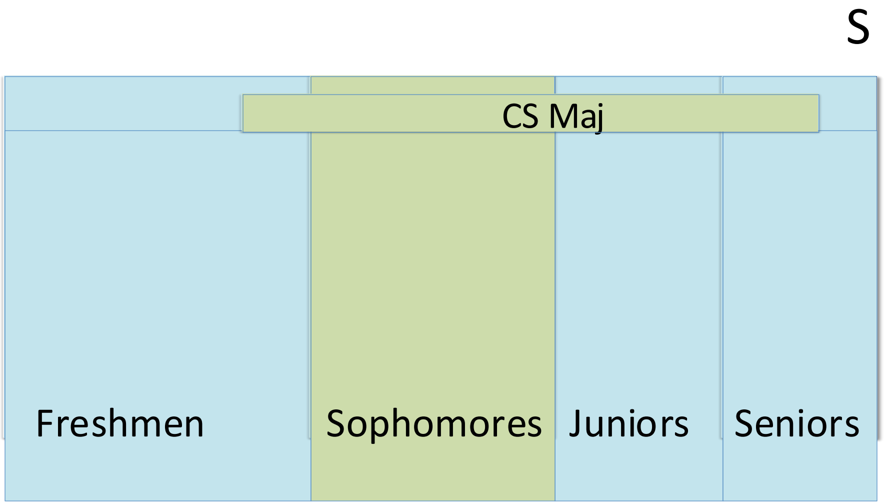
Probability as Area

P(Soph) + P(CS)

= P(CS , Soph.) + P(Non-CS , Soph.) +

P(Fr. CS) + P(Soph. CS) + P(J. CS) + P(Sen. CS)

Soph. CS is counted twice



So, P(Soph OR CS)

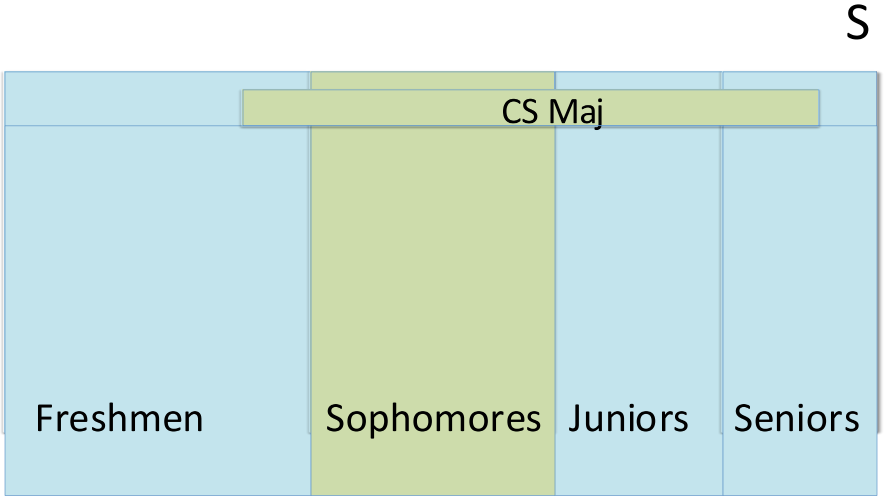
= P(Soph) + P(CS) - P(Soph. CS)

Inclusion-Exclusion Principle

Inclusion-Exclusion Principle For any events E and F,

$$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$$

Accounting for overlap between E and F



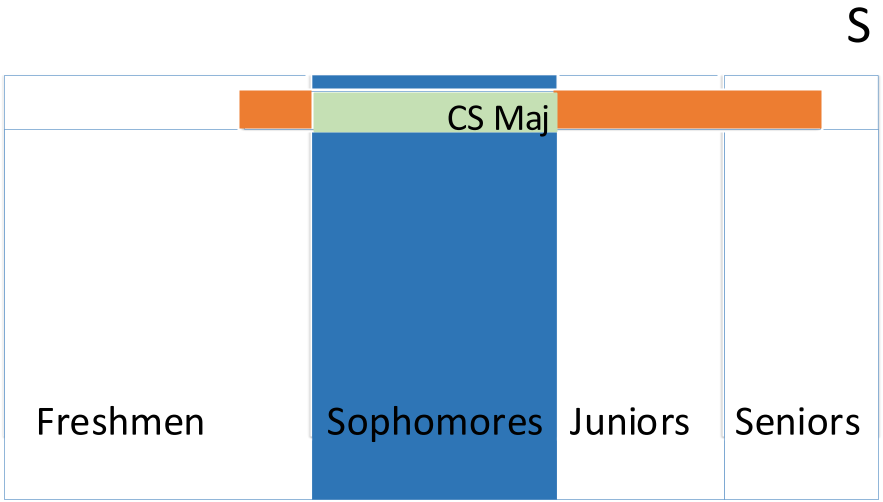
Recap

- Outcome: a single observation
- Sample space S : the set of all possible outcomes
- Event: a set of outcomes, a subset of sample space
- Disjoint events:
  - cannot happen together
  - $P(E \text{ } or \text{ } F) = P(E) + P(F)$
- Law of total probability: If disjoint events $E_1, \dots, E_n$ forms a partition of S , for event F(law of total probability):

$$P(F) = P(E_1, F) + P(E_2, F) + \cdots + P(E_n, F)$$

Inclusion-Exclusion Principle

- P(Soph) = P(CS , Soph.) + P(Non-CS , Soph.)
- P(CS) = P(Fr. CS) + P(Soph. CS) + P(J. CS) + P(Sen. CS)
- P (Soph or CS) = P(Soph) + P(CS) – P (Soph, CS)



Complementary events

How would I find *P*(Non-Sophomore)?

- Could just list the non-sophomores and then count, but we can use the fact that *P*(*S*) = 1 and *subtract* instead.
  - *P*(Non-Sophomore) = 1 − *P*(Sophomore)

Set operations

Events and Set Theory

An event is a set of outcomes and a subset of sample space, so we use set theory to describe and combine them.

Two dice example:

E_1 : *First* die rolls 1

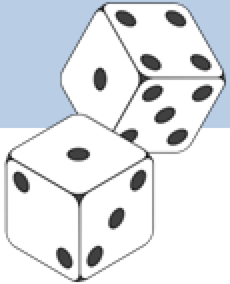
$E_1 = \{(1,1),(1,2),\ldots,(1,6)\}$

E_2 : *Second* die rolls 1

$E_2 = \{(1,1),(2,1),\ldots,(6,1)\}$

- Any die rolls 1: event $E_1\, or\, E_2$
- Set operation: $E_1 \cup E_2$

Set operations



Two dice example:

E_1 : *First* die rolls 1 E_2 : *Second* die rolls 1

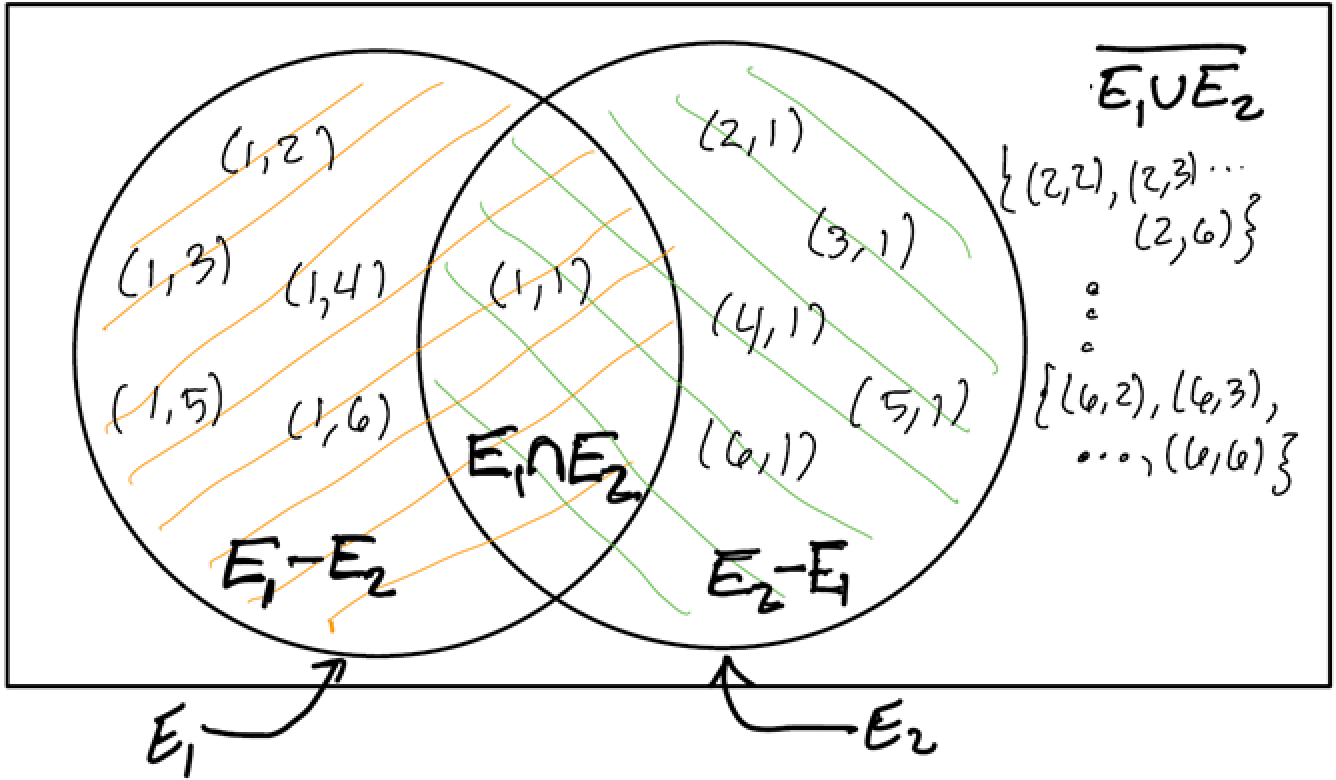
$$E_1 = \{(1,1),(1,2),\ldots,(1,6)\} \qquad E_2 = \{(1,1),(2,1),\ldots,(6,1)\}$$

Operators on events:

Operation	Value	Interpretation
$E_1 \cup E_2$	$\{(1,1),(1,2),\ldots,(1,6),(2,1),\ldots,(6,1)\}$	Any die rolls 1
$E_1 \cap E_2$	$\{(1,1)\}$	Both dice roll 1
$E_1 \setminus E_2$ $(= E_1 - E_2 := E_1 \cap E_2^c)$	$\{(1,2),(1,3),(1,4),(1,5),(1,6)\}$	Only the first die rolls 1
$\overline{E_1 \cup E_2}$ $(= (E_1 \cup E_2)^c)$	$\{(2,2),(2,3),\ldots,(2,6),(3,2),\ldots,(6,6)\}$	No die rolls 1

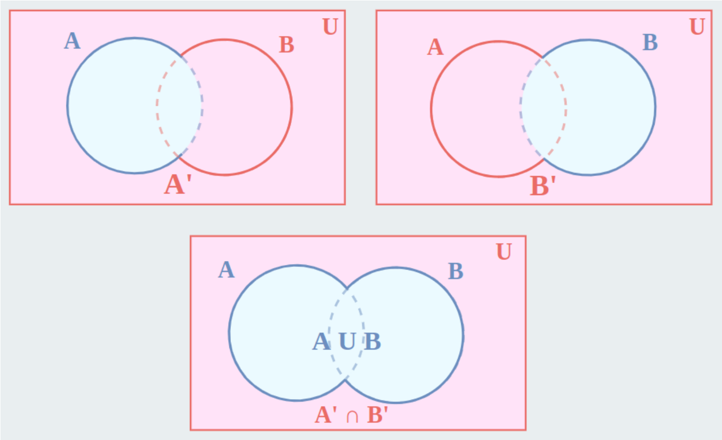
Set operations

Can interpret these operations using a Venn diagram…



Set Theory: De Morgan Law

De Morgan Law 1 $(A \cup B)^C = A^C \cap B^C$



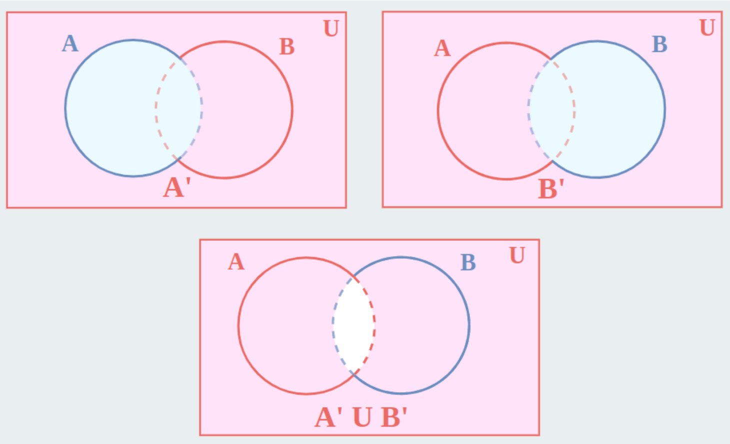
Example:

- A: I bring my cellphone
- B: I bring my laptop
- A^C: I don't bring my cellphone
- B^C: I don't bring my laptop

- $A \cup B$: I bring my cellphone or my laptop
- $(A \cup B)^C$: I bring neither my cellphone nor my laptop
- $A^C \cap B^C$: I didn't bring my cellphone & I didn't bring my laptop

Set Theory: De Morgan Law

De Morgan Law 2 $(A \cap B)^C = A^C \cup B^C$

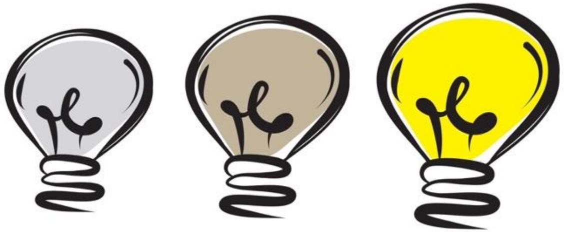


Example:

- A: I bring my cellphone
- B: I bring my laptop
- A^C: I don't bring my cellphone
- B^C: I don't bring my laptop

- $A \cap B$:
- $(A \cap B)^C$:
- $A^C \cup B^C$:

Intersection / union over n events



- n lightbulbs
- E_i: i-th lightbulb is on

- How to describe the event that at least one lightbulb is on?
  - i.e. bulb 1 is on OR … OR bulb n is on

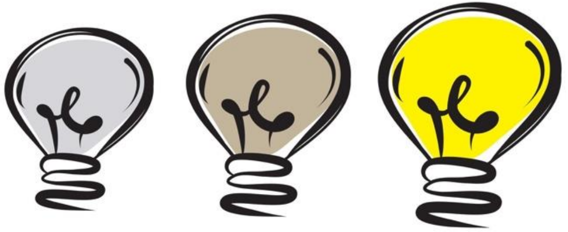
$$E_1 \cup \cdots \cup E_n =: \cup_{i=1}^{n} E_i$$

- How to describe the event that all lightbulbs are on?

$$E_1 \cap \cdots \cap E_n = \cap_{i=1}^{n} E_i$$

De Morgan Laws with n events

- **De Morgan Laws:**



$$(E_1 \cup \cdots \cup E_n)^C = E_1^C \cap \cdots \cap E_n^C$$

Not (at least one bulb is on) All bulbs are off

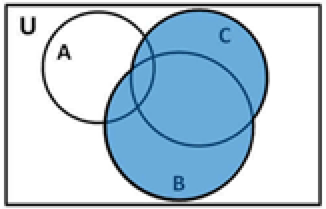
$$(E_1 \cap \cdots \cap E_n)^C = E_1^C \cup \cdots \cup E_n^C$$

Not (all bulbs are on) At least one bulbs is off

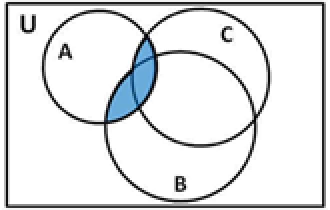
Set operation: Distributive law

- Distributive law $a(x + y) = ax + ay$ carry over to sets
- **Distributive Law 1** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

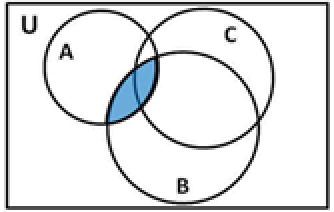
Draw your Venn diagram



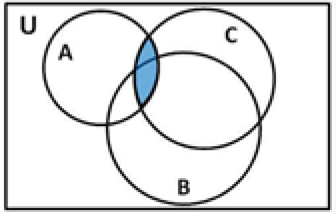
(B ∪ C)



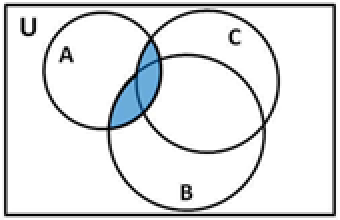
A ∩ (B ∪ C)



(A ∩ B)



(A ∩ B)



(A ∩ B) ∪ (A ∩ C)

Set operation: Distributive law

- **Distributive Law 2** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

- Can justify this by:
  - drawing a picture (like previous slide), or
  - proving it using Distributive Law 1 and De Morgan Law

Rules of Probability

Rules of probability

- To recap and summarize:

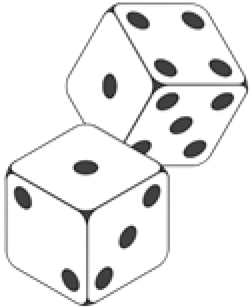
Rules of Probability

1. **Non-negativity:** All probabilities are between 0 and 1 (inclusive)
2. **Unity of the sample space:** $P(S) = 1$
3. **Complement Rule:** $P(E^C) = 1 - P(E)$
4. **Probability of Unions:**
   (a) *In general,* $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
   (b) *If E and F are disjoint, then* $P(E \cup F) = P(E) + P(F)$

Classical probability model

Special case

Assume each outcome is equally likely, and sample space is finite, then the probability of event is:



$$P(E) = \frac{|E|}{|S|}$$

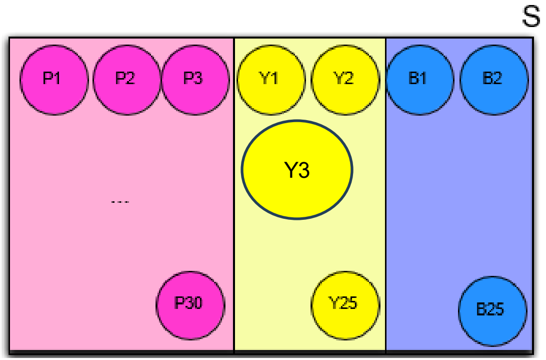
Number of elements in event set

Number of possible outcomes (e.g. 36)

This is called classical probability model

Rethinking the classical probability model

- Classical probability model assumes all outcomes equally likely
- When is this applicable?
  - *Fair* coin toss, *fair* dice throw, …
  - S = {P1, P2, …, P30, Y1, …, Y25, B1, …, B25}
- When is this assumption problematic?
  - *Unfair* coin toss (one side is heavier)
  - A yellow ball is much larger



Exercise: Blood types

- Human blood is classified by the presence or absence of two antigens, called A and B.
- If A is the event "presence of antigen A", and B is the event "presence of antigen B", what is:
  - $P(A \cap B)$? What is this event in words?

		Antigen B	
		Absent	Present
Antigen A	Absent	0.44	0.10
	Present	0.42	0.04

Exercise: Blood types

		Antigen B	
		Absent	Present
Antigen A	Absent	0.44	0.10
	Present	0.42	0.04

$(A \cup B)^C = A^C \cap B^C$
$A \cup B = ((A \cup B)^C)^C = (A^C \cap B^C)^C$

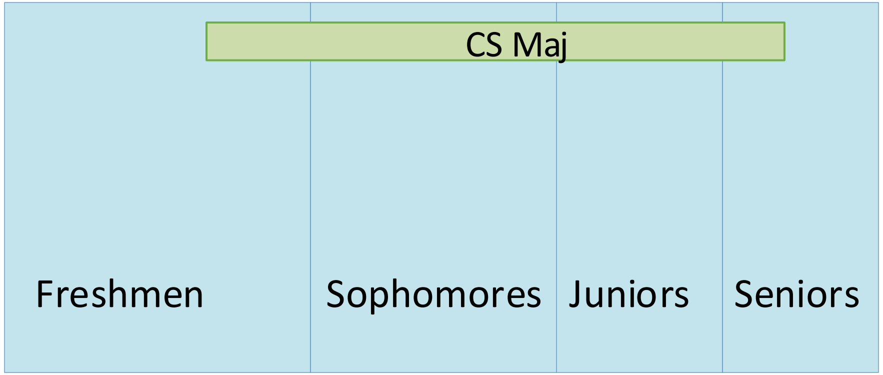
- What is $P(A \cup B)$ using De Morgan's Law?
- Rephrase $A \cup B$:
  - $A \cup B = (A^C \cap B^C)^C$, by De Morgan's Law
- Use the Complement Rule:
  - $P(A \cup B) = 1 - P(A^C \cap B^C) = 1 - 0.44 = 0.56$

Law of Total Probability

Law of Total Probability

- We saw that:

$$P(\text{CS}) = P(\text{Fr.}, \text{CS}) + P(\text{Soph.}, \text{CS}) + P(\text{J.} \, \text{CS}) + P(\text{Sen.} \, \text{CS})$$

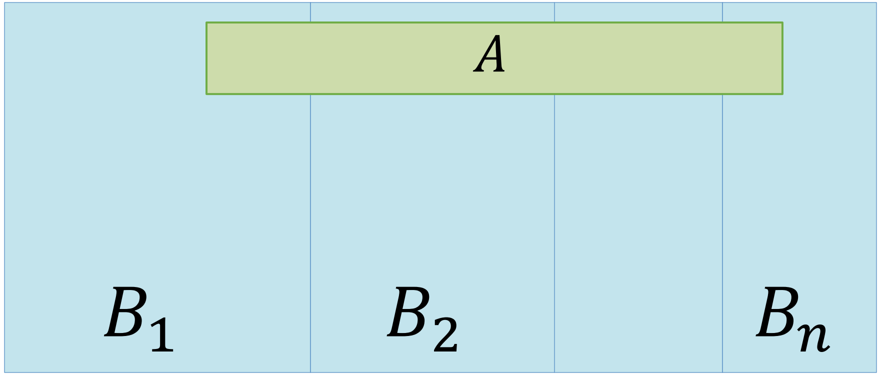


- Would the equality still be true if, say, we drop $P(\text{Sen.} \, \text{CS})$?
  - No – the three remaining events no longer form a partition of {CS}

Law of Total Probability

Law of Total Probability Suppose $B_1, \ldots, B_n$ *form a partition* of the sample space S. Then,

$$P(A) = P(A, B_1) + \cdots + P(A, B_n)$$



- Recall notation: $P(A, B_1)$ is a shorthand for $P(A \cap B_1)$

Law of Total Probability: blood types

		Antigen B Absent	Antigen B Present	Marginal
Antigen A	Absent	0.44	0.10	0.54
	Present	0.42	0.04	0.46
	Marginal	0.86	0.14	1.00

- B, B^C form a partition of sample space S, so
- $P(A) = P(A, B) + P(A, B^C) =$ 0.04+ 0.42 = 0.46
- Likewise,
- $P(B) = P(B, A) + P(B, A^C) =$ 0.04 + 0.10 = 0.14

Law of Total Probability: blood types

		Antigen B Absent	Antigen B Present	Marginal
Antigen A	Absent	0.44	0.10	0.54
	Present	0.42	0.04	0.46
	Marginal	0.86	0.14	1.00

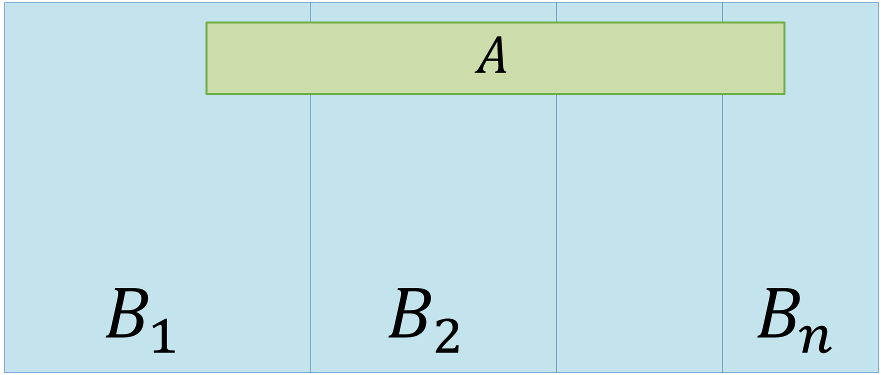
- Calculate $P(A \cup B)$ using inclusion-exclusion principle and law of total probability:
- $P(A \cup B) = P(A) + P(B) - P(A, B) = 0.46 + 0.14 - 0.04 = 0.56$

Law of Total Probability: another example

Example Roll two fair dice. Let X be the outcome of the first die. Let Y be the sum of both dice. What is the probability that both dice sum to 6 (i.e., Y=6)?

$$p(Y = 6) = \sum_{x=1}^{6} p(Y = 6, X = x)$$

$\{X = 1\} \ldots, \{X = 6\}$ form a partition of sample space S



$$= p(Y = 6, X = 1) + p(Y = 6, X = 2) + \ldots + p(Y = 6, X = 6)$$
$$= \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + 0 = \frac{5}{36}$$

Summary: calculating probabilities

- If we know that all outcomes are equally likely, we can use

We will use combinatorics to do counting

$$P(E) = \frac{|E|}{|S|}$$

Number of elements in event set ($|E|$)

Number of possible outcomes (e.g. 36) ($|S|$)

- If |E| is hard to calculate directly, we can try
  - the rules of probability
  - the Law of Total Probability, using an appropriate partition of sample space S