Computer Science

CSC380: Principles of Data Science

Probability 2

Xinchen Yu

Rules of probability

- To recap and summarize:

Rules of Probability

1. **Non-negativity:** All probabilities are between 0 and 1 (inclusive)
2. **Unity of the sample space:** $P(S) = 1$
3. **Complement Rule:** $P(E^C) = 1 - P(E)$
4. **Probability of Unions:**
   (a) *In general,* $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
   (b) If E and F are disjoint, then $P(E \cup F) = P(E) + P(F)$

Summary: calculating probabilities

- If we know that all outcomes are equally likely, we can use

We will use combinatorics to do counting

$$P(E) = \frac{|E|}{|S|}$$

Number of elements in event set

Number of possible outcomes (e.g. 36)

- If |E| is hard to calculate directly, we can try
  - the rules of probability
  - the Law of Total Probability, using an appropriate partition of sample space S

Overview

- Conditional probability
- Probabilistic reasoning
    - contingency table
    - probability trees

Conditional Probability

Example: Seat Belts

		Child Buck.	Child Unbuck.	Marginal
Parent	Buck.	0.48	0.12	0.60
	Unbuck.	0.10	0.30	0.40
	Marginal	0.58	0.42	1.00

Table: Probability Estimates for Seat Belt Status

Suppose we pick a family from US at random:

- What is the probability of the event “Child is Buckled”?
- What should our new estimate be if we know that “Parent is Buckled”?

Example: blood types

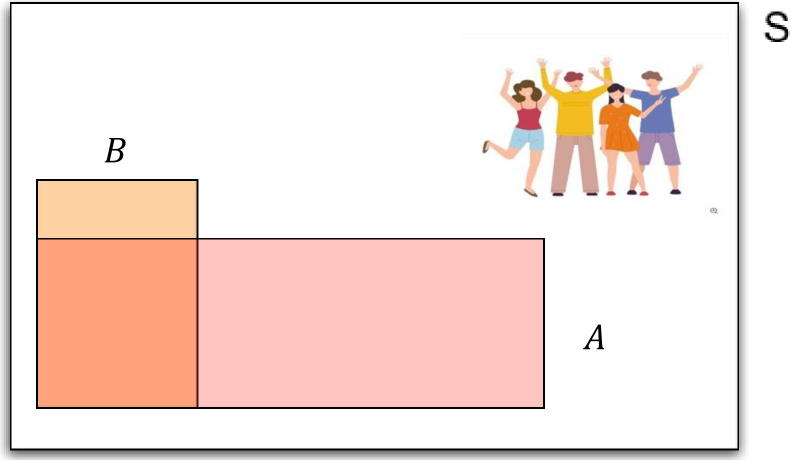
		Antigen B Absent	Antigen B Present	Marginal
Antigen A	Absent	0.44	0.10	0.54
	Present	0.42	0.04	0.46
	Marginal	0.86	0.14	1.00

Table: Probability Estimates for U.S. Blood Types

- A: "presence of antigen A", B: "presence of antigen B"
- Suppose someone of an unknown blood type gets a test that reveals the presence of antigen A. What is the chance that:
    - event A happens to them?
    - event B happens to them?

Relative area

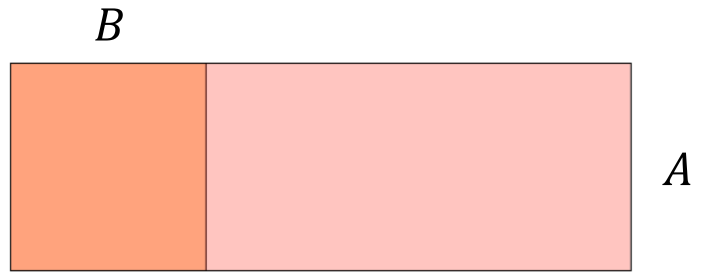
- A: antigen A present B: antigen B present
- Given that A happens, what is the chance of B happening?



- Restricted to people with antigen A present, what is the fraction of those people with antigen B?

Relative area

- Let's zoom into people with antigen A present.

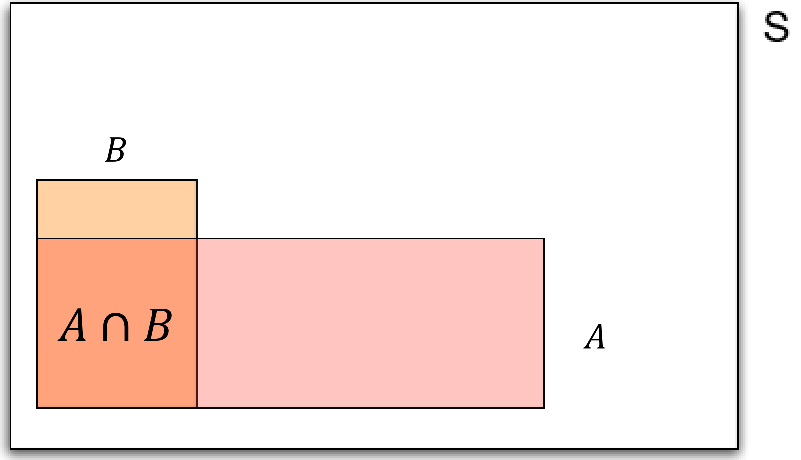


- It's just as if the sample space had shrunk to include only A
- Now, probabilities correspond to proportions of A
- What does the orange square represent?
  - $A \cap B$
- How would we find the probability of B given A?

Conditional Probability

- To find the conditional probability of B given A, consider the ways B can occur in the context of A (i.e., $A \cap B$), out of all the ways A can occur:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$



Example:
A: currently inside a cafe
B: drinking coffee right now

Conditioning changes the sample space

- Before we knew anything, anything in sample space S could occur.
- After we know A happened, we are only choosing from within A.
- The set A becomes our new sample space
- Instead of asking "In what proportion of S is B true?", we now ask "In what proportion of A is B true?"

For example, rolling a fair die, define A: even numbers, B: get a 2.

- Before knew anything, P(B) is 1/6
- After knowing A, P(B) is (1/6) / (1/2) = 1/3

Every Probability is a Conditional Probability

- We can consider the original probabilities to be conditioned on the event S: at first what we know is that "something in S" occurs.

$$P(B) = P(B|S)$$

$$P(B \cap C) = P(B \cap C|S)$$

$$P(\,B \mid S\,) = \frac{P(B \cap S)}{P(S)} = P(B)$$

- $P(B|S)$ in words: what proportion of S does B happen?
- If we then learn that A occurs, A becomes our restricted sample space.
- $P(B|A)$ in words: what proportion of A does B happen?

Joint Probability and Conditional Probability

- We can rearrange $P(\,B \mid A\,) = \frac{P(A \cap B)}{P(A)}$ and derive:

The "Chain Rule" of Probability

For any events, A and B, the joint probability $P(A \cap B)$ can be computed as

$$P(A \cap B) = P(B|A) \times P(A)$$

Or, since $P(A \cap B) = P(B \cap A)$

$$P(A \cap B) = P(A|B) \times P(B)$$

Terminology

When we have two events A and B…

- Conditional probability: $P(A|B), P(A^c|B), P(B|A)$ etc.
- Joint probability: $P(A, B)$ or $P(A^c, B)$ or ...
- Marginal probability: $P(A)$ or $P(A^c)$

Example revisited: blood types

		Antigen B Absent	Antigen B Present	Marginal
Antigen A	Absent	0.44	0.10	0.54
	Present	0.42	0.04	0.46
	Marginal	0.86	0.14	1.00

Table: Probability Estimates for U.S. Blood Types

- Suppose someone of an unknown blood type gets a test that reveals the presence of antigen A.
  - What is $P(\,A \mid A\,)$?

$$P(\,A \mid A\,) = \frac{P(A \cap A)}{P(A)} = 1$$

  - What is $P(\,B \mid A\,)$?

$$P(\,B \mid A\,) = \frac{P(A \cap B)}{P(A)} = \frac{0.04}{0.46} = 0.087$$

Example revisited: Seat Belts

A: parent is buckled
C: child is buckled

		Child Buck.	Child Unbuck.	Marginal
Parent	Buck.	0.48	0.12	0.60
	Unbuck.	0.10	0.30	0.40
	Marginal	0.58	0.42	1.00

Table: Probability Estimates for Seat Belt Status

Suppose we pick a family from US at random:

- What is the probability of the event "Child is Buckled"? $P(C)$
- What should our new estimate be if we know that ("given that") Parent is Buckled? $P(C \mid A)$

Example revisited: Seat Belts

A: parent is buckled
C: child is buckled

		Child Buck.	Child Unbuck.	Marginal
Parent	Buck.	0.48	0.12	0.60
	Unbuck.	0.10	0.30	0.40
	Marginal	0.58	0.42	1.00

Table: Probability Estimates for Seat Belt Status

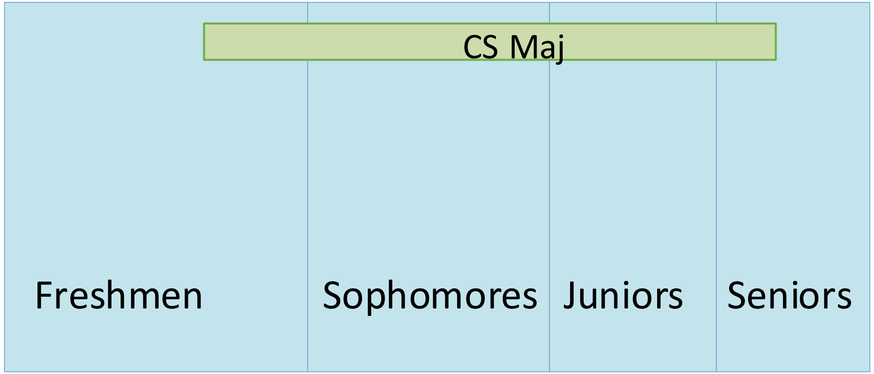
Suppose we pick a family from the US at random:

- $P(C) = 0.58$
- $P(\,C \mid A\,) = \frac{P(C \cap A)}{P(A)} = \frac{0.48}{0.60} = 0.8$ Larger than $P(C)$
- Suppose we see a buckled parent, it is much more likely that we see their child buckled

Law of Total Probability, revisited

Law of Total Probability Suppose $B_1, \ldots, B_n$ *form a partition* of the sample space S. Then,

$$P(A) = P(A, B_1) + \cdots + P(A, B_n)$$



Law of Total Probability, revisited

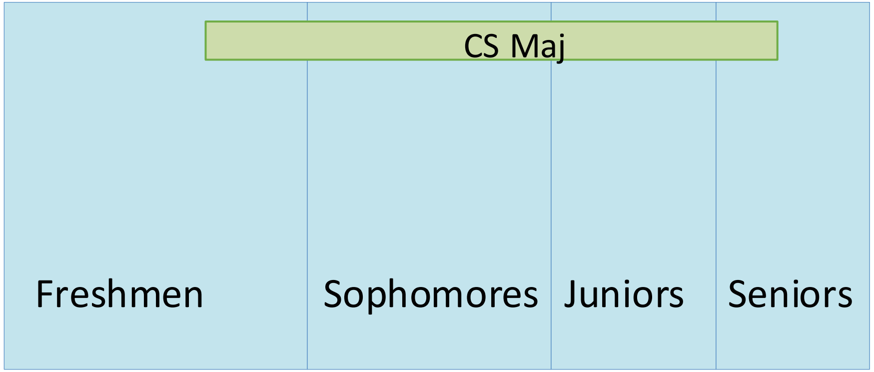
Expanding each $P(A, B_i) = P(A \mid B_i)P(B_i)$, we have:

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i)P(B_i)$$

A: student in CS major

B_i : student in class year i

$P(A \mid B_i)$ The fraction of CS major in class year i



Law of Total Probability, revisited

Example Suppose UA has an equal number of students in the 4 class years, and the fraction of CS major in these 4 class years are 10%, 10%, 20%, 80% respectively. What is fraction of CS majors?

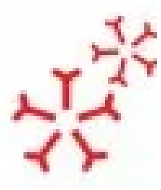
- $P(B_1) = P(B_2) = P(B_3) = P(B_4) = 0.25$
- $P(\, C \mid B_1 \,) = 0.1, \ldots, P(\, C \mid B_4 \,) = 0.8$
- Calculate $P(C)$ by:

$$P(C) = \sum_{i=1}^{4} P(\, C \mid B_i \,) P(B_i) = 30\%$$

Probabilistic reasoning

Probabilistic reasoning

- We have some prior belief of an event A happening
  - $P(A)$, prior probability
  - e.g. me infected by COVID



- We see some new evidence B
  - e.g. I test COVID positive

- How does seeing B affect our belief about A?
  - $P(A \mid B)$, posterior probability

Another example: detector

A store owner discovers that some of her employees have taken cash. She decides to use a detector to discover who they are.

- Suppose that 10% of employees stole.
- The detector buzzes 80% of the time that someone stole, and 20% of the time that someone not stole
- Is the detector reliable? In other words, if the detector buzzes, what’s the probability that the person did stole?

H: employee not stole

B: lie detector buzzes

Another example: detector

- Suppose that 10% of employees stole.

  H: employee did not stole $P(H) = 0.9$

- The detector buzzes 80% of the time that someone stoles, and 20% of the time that someone not stole.

  B: lie detector buzzes

  $$P(B \mid H^C) = 0.8$$

  $$P(B \mid H) = 0.2$$

- If the detector buzzes, what's the probability that the person stole?

$$P(H^C \mid B)$$

Detector analysis: Probability table

		Detector result		
		Pass (B^C)	Buzz (B)	Marginal
Employee	Not stole (H)			
	Stole (H^C)			
	Marginal			

$P(H) = 0.9$

$P(B \mid H^C) = 0.8$

$P(B \mid H) = 0.2$

Detector analysis: Probability table

$$P(H, B) = P(H) \cdot P(B \mid H) = 0.9 \times 0.2 = 0.18$$

		Detector result		
		Pass (B^C)	Buzz (B)	Marginal
Employee	Not stole (H)		0.18	0.9
	Stole (H^C)			0.1
	Marginal			

$$P(H) = 0.9$$

$$P(B \mid H^C) = 0.8$$

$$P(B \mid H) = 0.2$$

Detector analysis: Probability table

$$P(H) = P(H, B) + P(H, B^c) = 0.9$$

		Detector result		
		Pass (B^C)	Buzz (B)	Marginal
Employee	Not stole (H)	0.72	0.18	0.9
	Stole (H^C)			0.1
	Marginal			

$$P(H) = 0.9$$

$$P(B \mid H^C) = 0.8$$

$$P(B \mid H) = 0.2$$

Detector analysis: Probability table

		Detector result		
		Pass (B^C)	Buzz (B)	Marginal
Employee	Not stole (H)	0.72	0.18	0.9
	Stole (H^C)	0.02	0.08	0.1
	Marginal	0.74	0.26	1

$P(H) = 0.9$

$P(B \mid H^C) = 0.8$

$P(B \mid H) = 0.2$

Detector analysis: Probability table

		Detector result		
		Pass (B^C)	Buzz (B)	Marginal
Employee	Not stole (H)	0.72	0.18	0.9
	Stole (H^C)	0.02	0.08	0.1
	Marginal	0.74	0.26	1

- We have the full probability table. Can we calculate $P(H^C \mid B)$? Yes!

$$P(H^C \mid B) = \frac{P(H^C, B)}{P(B)} \qquad \frac{0.08}{0.26} = 0.307$$

It seems like the detector is not very reliable…

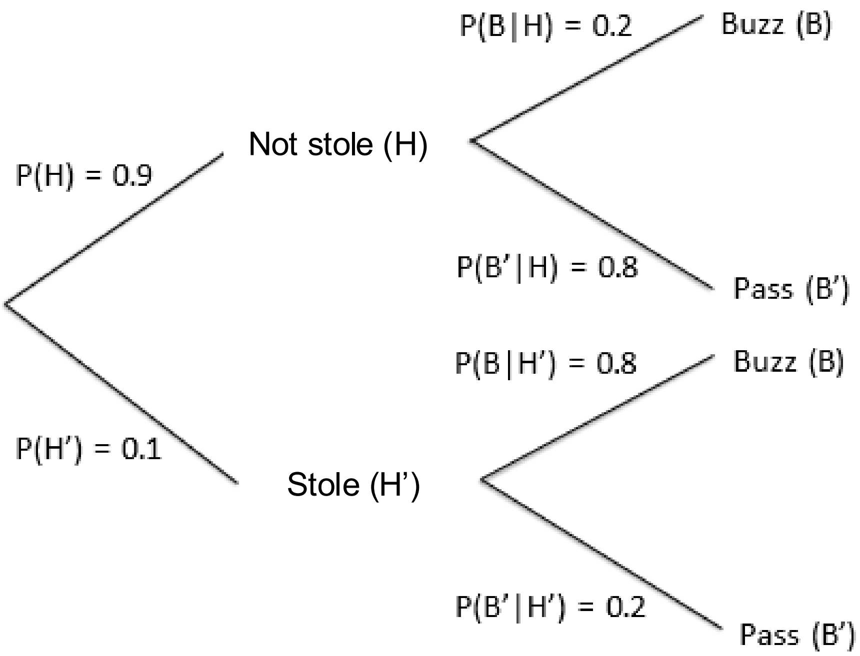
Recap

- Conditional probability: $P(\, B \mid A\,) = \frac{P(A \cap B)}{P(A)}$
- Law of total probability: $P(A) = \sum_{i=1}^{n} P(A, B_i) = \sum_{i=1}^{n} P(\, A \mid B_i\,) P(B_i)$
- If we know $\mathrm{P}(H), P(B|H^C), P(B|H)$:
  - $\mathrm{P}(H) \to P(H^C)$ Complement rule
  - $\mathrm{P}(H), P(B|H) \to P(B, H)$ joint probability
  - $\mathrm{P}(H^C), P(B|H^C) \to P(B, H^C)$ joint probability
  - $\mathrm{P(B)} \to P(B, H) + P(B, H^C)$ marginal probability
  - $P(B), P(B, H) \to P(H|B)$ conditional probability
  - $P(B), P(B, H^C) \to P(H^C|B)$ conditional probability
- We can get $P(B), P(H|B), P(H^C|B)$

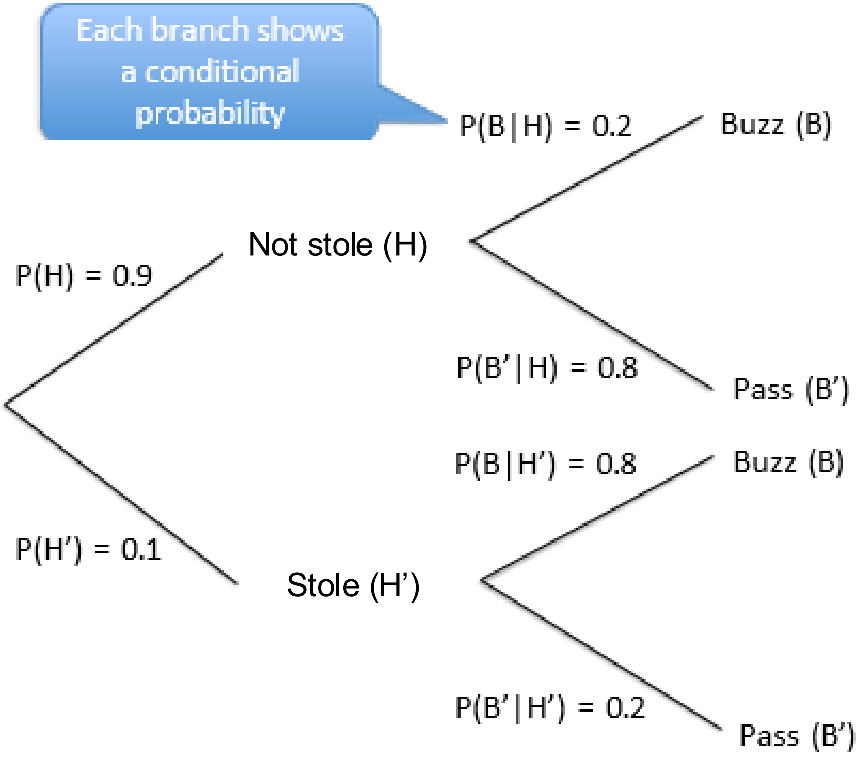
Today’s plan

- Another tool: probability trees
- Bayes rule
- Bayes rule and law of total probability

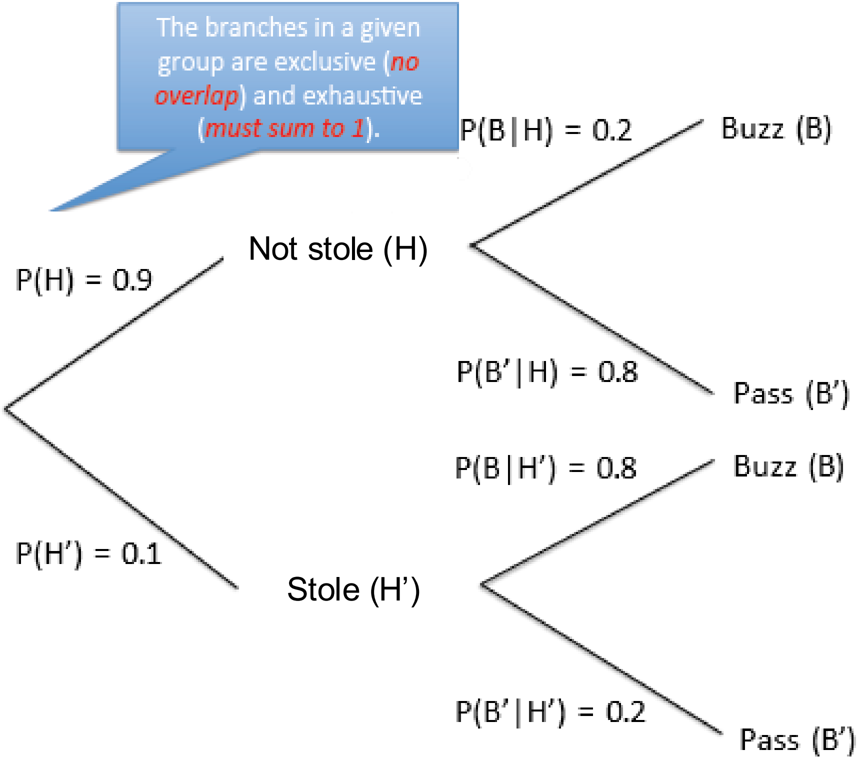
Probability trees: another useful tool



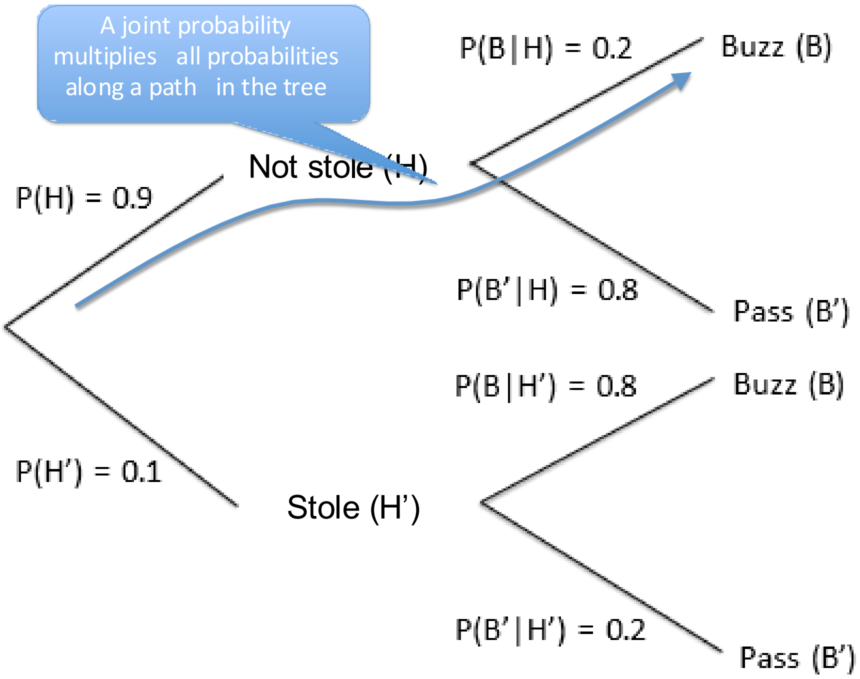
Probability trees: another useful tool



Probability trees: another useful tool

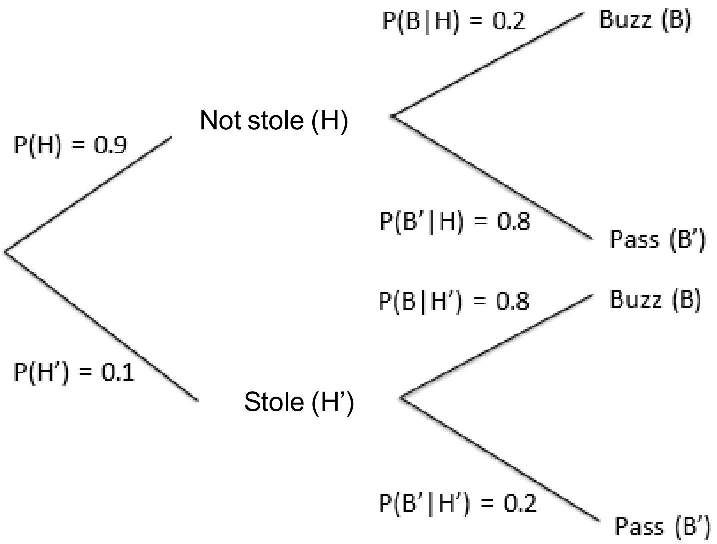


Probability trees: another useful tool



Probability trees: another useful tool

- What is $P(\text{Buzz}, \text{Stole})$?
  - 0.08
- $P(\text{Buzz})$?
  - Hint: which branches end up with buzzing?
  - 0.26 (0.08+0.18)
- $P(\text{Stole} \mid \text{Buzz})$?
  - Hint: which of the prev. branches contains the stole event?
  - 0.08 / 0.26



In-class activity: COVID test

The Public Health Department gives us the following information:

- A test for the disease yields a positive result (+) 90% of the time when the disease is present (Y) $P(+ \mid Y) = 0.9$, "sensitivity" of the test

- A test for the disease yields a positive result 1% of the time when the disease is not present (N) $P(+ \mid N) = 0.01$

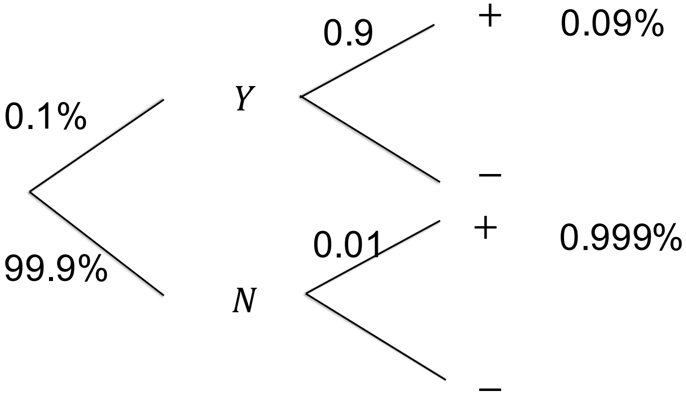
- One person in 1,000 has the disease. $P(Y) = 0.1\%$

<u>Draw a probability tree and use it to answer:</u> what is the probability that a person with positive test has the disease?

$P(Y \mid +)$?

In-class activity: COVID test

- Goal: calculate P(Y | +)
- Two branches are associated with positive test results +
  - What are the associated events?



- $P(+,Y) = P(+ | Y)P(Y) = 0.09\%$
- $P(+,N) = P(+ | N)P(N) = 0.999\%$

P(+ | Y) = 0.9
P(+ | N) = 0.01
P(Y) = 0.001

- $P(Y \mid +) = \frac{P(+,Y)}{P(+)} = \frac{0.09\%}{0.09\% + 0.999\%} \approx \frac{1}{12}$
- Conclusion: being tested positive does not mean much..

In-class activity: COVID test

Probabilistic reasoning tells as how does seeing a new evidence affect our prior belief about an event.

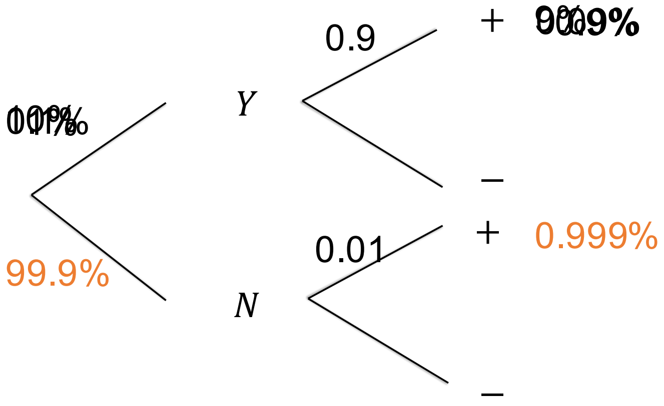
- Prior probability: one person in 1,000 has the disease: P(Y) = 0.1%
- New evidence: seen a person is tested positive
- Posterior probability: a person with positive test has the disease:

$$P(Y \mid +) = \frac{0.09\%}{0.09\% + 0.999\%} \approx \frac{1}{12}$$

COVID test: additional insights

- What would P(Y | +) look like, if instead:
  - 1 in 100 people have COVID?
  - 1 in 10?

$$P(Y \mid +) = \frac{P(+,Y)}{P(+)} = \frac{0.09\%}{0.09\% + 0.999\%} \approx \frac{1}{12}$$



Note: this branch's value is imprecise, but roughly stays close to 1%!

- Insight: *base rate* $P(Y)$ significantly affects P(Y | +), hence the conclusions we draw

Conditional probability: additional note

- The rules of probability also applies to the rules of conditional probability
- Just replace $P(E), P(F)$ with $P(E|A), P(F|A)$
  - But, need to condition on the same A in the same equation

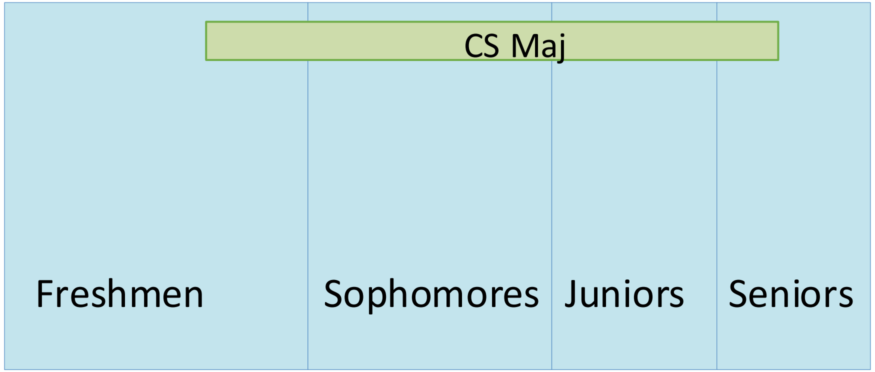
Rules of Probability

1. **Non-negativity:** All probabilities are between 0 and 1 (inclusive)
2. **Unity of the sample space:** $P(S) = 1$
3. **Complement Rule:** $P(E^C) = 1 - P(E)$
4. **Probability of Unions:**
   (a) *In general,* $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
   (b) If E and F are disjoint, then $P(E \cup F) = P(E) + P(F)$

Some examples

A: CS major

- $P(S|A) = 1$
- $P(E|A) + P(E^C|A) = 1$
- $P(E|A) + P(F|A) = P(E \cup F|A)$ for disjoint E and F



Bayes rule

Reversing conditional probabilities

- Is $P(A \mid B) = P(B \mid A)$ in general?
- Let's see..

$$P(A, B) = P(A \mid B) \cdot P(B) = P(B \mid A) \cdot P(A)$$

  - Equal only when $P(A)$ and $P(B)$ are equal

- Let's take a look at a real-world example when they are unequal…

Reversing conditional probabilities

Q: Hearing a French accent means someone is French?

Event A: A person is from France.

Event B: A person speaks English with a French accent.

- In a diverse city, only 5% of people are from France
- Of those from France, 80% speak English with a French accent: $P(B|A)$
- Of those not from France, only 2% speak English with a French accent (due to schooling, mimicry, or neighboring countries)

What is $P(A), P(B)$ and $P(A|B)$?

Reversing conditional probabilities

What is $P(A), P(B)$ and $P(B|A)$?

- $P(A) = 0.05$
- $P(B) = P(A, B) + P(A^c, B) = P(B|A) \cdot P(A) + P(B|A^c) \cdot P(A^c) = 0.8 \cdot 0.05 + 0.02 \cdot (1 - 0.05) = 0.04 + 0.019 = 0.059$
- $P(A \mid B) = P(A, B)/P(B) = 0.04/0.059 \approx 0.678$

So $P(A) \neq \mathrm{P(B)}$, also hearing a French accent doesn't guarantee someone is French: a ~68% chance

Bayes rule

Bayes rule For events A, B,

$$P(A \mid B) = \frac{P(A) \cdot P(B \mid A)}{P(B)}$$

- Very easy to derive from the chain rule, so remember that first.
- Named after Thomas Bayes (1701-1761), English philosopher & pastor

Bayes rule

Bayes rule For events A, B,

Prior probability Support of evidence

$$P(A \mid B) = \frac{P(A) \cdot P(B \mid A)}{P(B)}$$

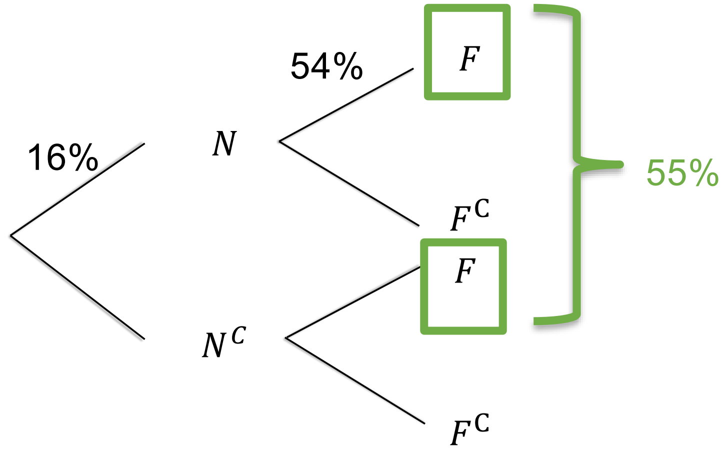
Posterior probability Probability of evidence

Examples:

- A: I have COVID, B: my test shows positive
- A: employee stole B: the detector buzzes
- A: student is CS major B: student is a senior

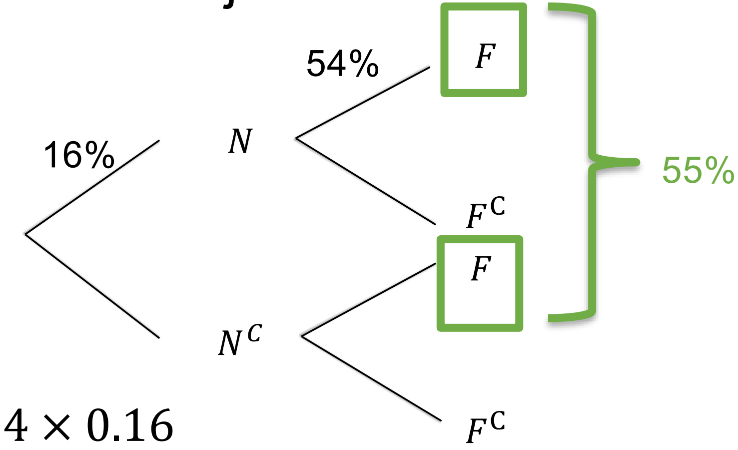
Bayes rule: another example

- In a class, 16% of the students are Nutrition Science majors, 55% students are female. Of the Nutrition Science majors, 54% are female.
- What proportion of female students in the class are Nutrition Science majors?
- What is the probability tree of this?
- We are looking for $P(N \mid F)$



Bayes rule: another example

- 16% of the students are Nutrition Science majors, 55% are female. Of the Nutrition Science majors, 54% are female. What proportion of female students in the class are Nutrition Science majors?



- We can use $P(N \mid F) = \frac{P(N,F)}{P(F)}$
  - We know $P(F) = 0.55$
- Can we obtain $P(N, F)$?
  - We can use $P(N, F) = P(F \mid N) \cdot P(N) = 0.54 \times 0.16$
- Altogether, we have

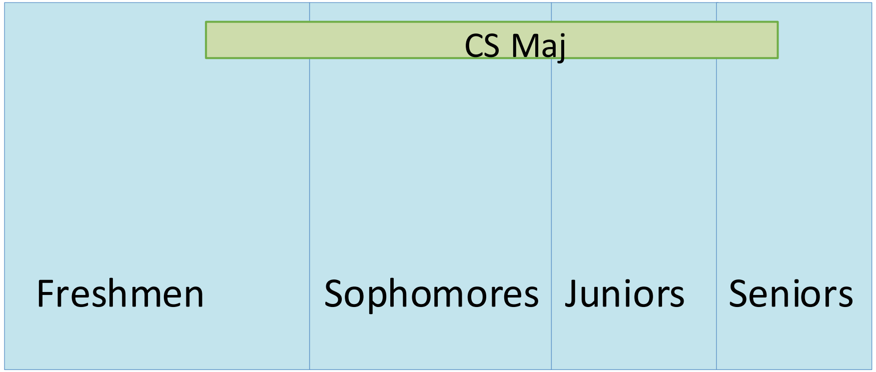
$$P(N \mid F) = \frac{P(F \mid N) \cdot P(N)}{P(F)} = \frac{0.54 \times 0.16}{0.55}$$

Bayes rule and Law of Total Probability

Bayes rule (equivalent form) For event A and $B_1, \ldots, B_n$ forming a partition of S,

$$P(B_i \mid A) = \frac{P(A \mid B_i) \cdot P(B_i)}{\sum_{j=1}^{n} P(A \mid B_j) \cdot P(B_j)}$$

← $P(A)$



....

Bayes rule and Law of Total Probability

Example Suppose UA has an equal number of students in the 4 class years, and the fraction of CS major in these 4 class years are 10% ($P(C|B_1)$), 10%, 20%, 80% respectively.

We have previously calculated that $P(C) = 30\%$

If we see a CS major student, what is their most likely year class?

$P(B_1 \mid C), \ldots, P(B_4 \mid C)$ -> maximum?

Bayes rule and Law of Total Probability

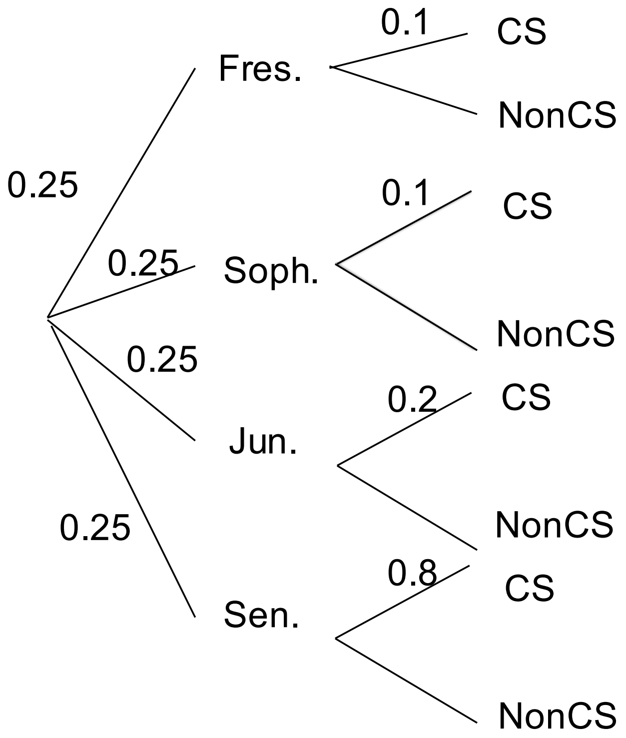
- Let's draw a probability tree..
- After learning that the student is CS major:

$$P(B_1 \mid C) = \frac{0.25 \times 0.1}{P(C)}$$

$$\cdots$$

$$P(B_4 \mid C) = \frac{0.25 \times 0.8}{P(C)}$$

- So most likely, this student is a senior
- Equivalent form: $P(B_i \mid C) \propto P(B_i)P(C \mid B_i)$
  - $\propto$: proportional to
  - $P(C)$ can be viewed as a *normalization factor*



Extension: chain rule for conditional probability

- If we deal with more than 3 events happening together, we can apply the chain rule of probability repeatedly:

Treat (B, C) as a single event

$$P(A, B, C) = P(A \mid B, C)\, P(B, C)$$

$$= P(A \mid B, C)\, P(B \mid C)\, P(C)$$

Independence

Probabilistic Independence

- Event S: 10% of employees stole.
- Event R: There's a 5% chance of rain tomorrow.
- What's the probability an employee stole if it rains tomorrow?

Probably your intuition is that one conveys no information about the other.
What does this mean about the relationship between P(R|S), and P(S)?

Probabilistic Independence

Independent Events

We say that event A is **independent** of event B if conditioning on B does not change the probability of A, that is if

$$P(A|B) = P(A)$$

- Is the independence symmetric?
- In other words, if P(A|B) = P(A), is P(B|A) = P(A)?

Probabilistic Independence

- If A is independent of B, then $P(\ A \mid B\ ) = P(A)$. Is $P(B|A)$ also equal to $P(B)$?

- Using Bayes' rule, we have

$$P(B|A) = \frac{P(\ A \mid B\ )P(B)}{P(A)}$$

- So independence is indeed a symmetric notion

Independence: equivalent statement

- If A, B are independent, then their joint probability has a simple form:

$$P(A, B) = P(A \mid B)P(B)$$
$$= P(A) \cdot P(B)$$

- This is an equivalent characterization of independence

Independence (version 2)

If A and B are independent events, then

$$P(A \cap B) = P(A)P(B)$$

Independence of several events

- We can generalize the notion of independence from two events to more than two.
  - E.g. A: employee stole; B: rain tomorrow, C: stock price up

- Events $A_1, \dots, A_n$ are independent if for any subsets $A_{i_1}, \dots, A_{i_j}$,

$$P\left(A_{i_1}, \dots, A_{i_j}\right) = P\left(A_{i_1}\right) \cdot \dots \cdot P(A_{i_j})$$

Independence of several events

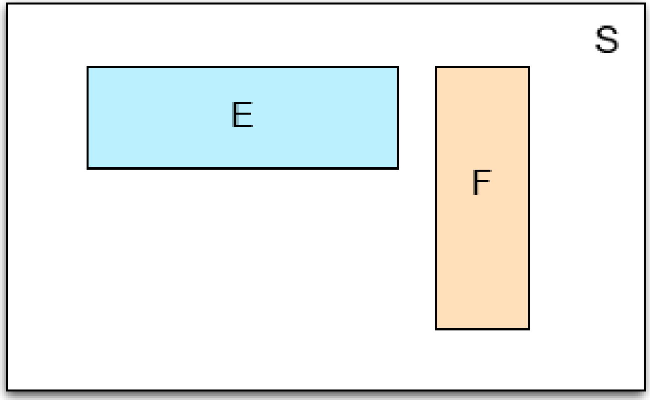
- If events A, B, C are independent, then
  - $P(A, B, C) = P(A) \cdot P(B) \cdot P(C)$
  - $P(A, C) = P(A) \cdot P(C)$
  - $P(B, C) = P(B) \cdot P(C)$

Rolling a die three times, the probability of sequence (1, 2, 3)?

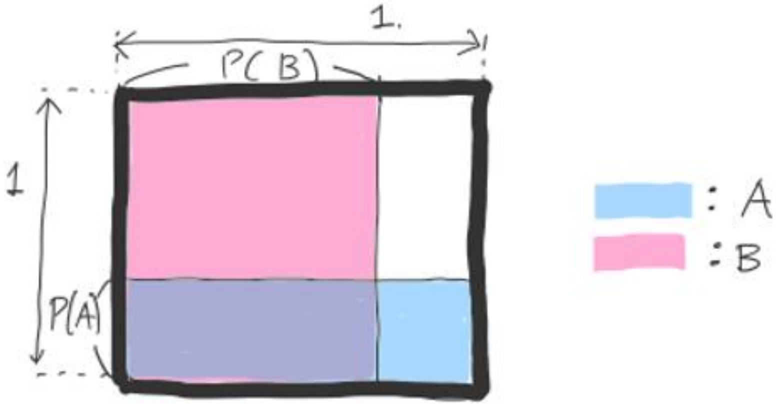
1/6 x 1/6 x 1/6

Independent vs. Disjoint Events

- Many people confuse independence with disjointness.
- They are very different!
- What do the Venn diagrams look like?



Disjoint



Independence: P(B|A) = P(B)

Independent vs. Disjoint Events

- If A and B are disjoint, what is P(B|A)?

$$P(\,B \mid A\,) = \frac{P(A,B)}{P(A)} = 0!$$

- Disjointness is practically the opposite of independence: if A occurs, B doesn't occur
- Defining property of independent events:

$$P(A \,\cap\, B) \,=\, P(A)P(B)$$

- Defining property of disjoint events:

$$P(A \,\cap\, B) \,=\, 0$$

Summary

Conditional Probability Summary

- Representing conditional probabilities using contingency tables, Venn diagrams, and probability trees.
- The chain rule
- Bayes rule
- The law of total probability
- Independent events
- Disjoint events

Probability and Combinatorics

Probability and Combinatorics

- Combinatorics (in CSc144) are useful in calculating probabilities
  - Permutations
  - Combinations
- Recall: when all outcomes are equally likely:

We will use combinatorics to do counting

$$P(E) = \frac{|E|}{|S|}$$

Number of outcomes in event set ($|E|$)

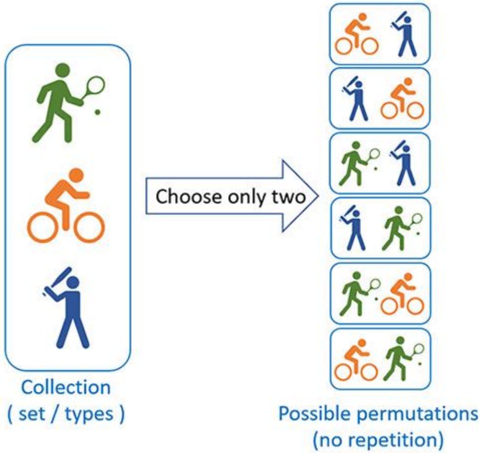
Number of possible outcomes (e.g. 36) ($|S|$)

Permutation number

- If *ordered* selection of k items out of n is done without replacement, there are

$$n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

outcomes



Choose 2 from 3 sports
for people A and B

Combination number

- If *unordered* selection of k items out of n is done without replacement, there are

$$\frac{n!}{(n-k)!\ k!} =: \binom{n}{k}$$

outcomes